Topological Quantum Error Correction with Optimal Encoding Rate

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We prove the existence of topological quantum error correcting codes with encoding rates k/n asymptotically approaching the maximum possible value. Explicit constructions of these topological codes are presented using surfaces of arbitrary genus. We find a class of regular toric codes that are optimal. For physical implementations, we present planar topological codes.

I. INTRODUCTION

Quantum computation has overcome major difficulties and has become a field of solid research. On the theoretical side, several models of quantum computation are already proposed like the quantum network model using a set of universal logic gates. Quantum error correction and fault tolerant quantum computation have been proved to be well established theoretically. On the experimental side, test-ground experiments have been conducted with a small number of quantum logic gates based on several proposals for realizing qubits. These constitute proof-of-principle experimental realizations showing that theory meets experiment.

Yet, it still faces a major challenge in order to built a real quantum computer: for a scalable quantum computer to be ever built, we have to battle decoherence and systematic errors in an efficient way [1], [2]. In fact, the network model corrects errors combinatorially and this requires a very low initial error rate, known as the threshold, in order to stabilize a quantum computation [3], [4], [5], [6].

There exists a very clever proposal of fault-tolerant quantum computation based on quantum topological ideas [7], [8]. The idea is to design the quantum operations so as to have a physically built-in mechanism for error correction, without resorting to external corrections every time an error occurs [9]. The key point here is that quantum topology is a global resource that is robust

FIG. 1: The rate k/n vs. t/n for the optimized toric codes ($\triangle$) and for the codes derived from self-dual embeddings of graphs K_{4l+1} ($\square$); $\bigcirc$ corresponds to the embedding of K_5 in the torus. The quantum Hamming bound is displayed as a reference (solid line).

against local errors, thereby providing a natural setup for fault tolerance [10].

II. TOPOLOGICAL CODES ON ARBITRARY SURFACES

A prerequisite for a topological QC is a topological quantum code for error detection and correction. It serves also as a quantum memory. In addition, quantum error correction codes are useful for quantum communication channels while sharing the feature of being quantumly robust.

A Hamiltonian can be constructed such that its ground state coincides with the code space. The nice thing of topological quantum codes (TQC) is that the generators are local and this makes feasible the experimental implementation of these codes, although other obstacles have to be overcome as we shall explain.

In this paper we shall provide explicit constructions of topological quantum codes with encoding rates beating those that can be achieved with current toric codes [7] (see Fig. 1 and eq. (10)).

A *quantum error correcting code of length n* is a subspace $\mathcal{C}$ of $\mathcal{H}_2^{\otimes n}$, with $\mathcal{H}_2$ the Hilbert space of one qubit, such that recovery is possible after noise consisting of any combination of error operators from some set $\mathcal{E}$ of operators on $\mathcal{H}_2^{\otimes n}$. The set $\mathcal{E}$ is the set of *correctable errors*, and we say that $\mathcal{C}$ *corrects* $\mathcal{E}$. For codes of length n, let $\mathcal{E}(n, k)$ be the set of operators acting on at most k qubits. We define the *distance* of the code $\mathcal{C}$, denoted $d(\mathcal{C})$, as the smallest number d for which the code does not detect $\mathcal{E}(n, d)$. A code $\mathcal{C}$ corrects $\mathcal{E}(n, t)$ iff $d(\mathcal{C}) > 2t$. In this case we say that $\mathcal{C}$ corrects t errors. We talk about $[[n, k, d]]$ codes when referring to quantum codes of length n, dimension 2^k and distance d. Such a code is said to encode k qubits. The *encoding rate* is $\frac{k}{n}$.

We consider the following family of operators acting on a string of qubits of length n:

$$\sigma_{\mathbf{v}} := \sigma_{\mathbf{xz}} := \bigotimes_{j=1}^{n} i^{x_j z_j} X^{x_j} Z^{z_j}, \tag{1}$$

where $\mathbf{x}, \mathbf{z} \in \mathbf{Z}_2^n$, $\mathbf{v} = (\mathbf{xz}) := (x_1, \ldots, x_n, z_1, \ldots, z_n)$ and X, Z are the standard Pauli matrices. They commute as

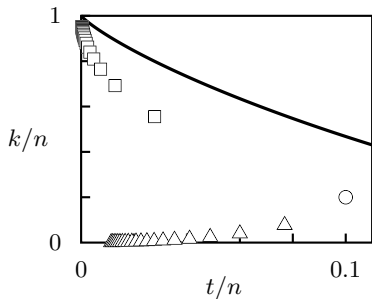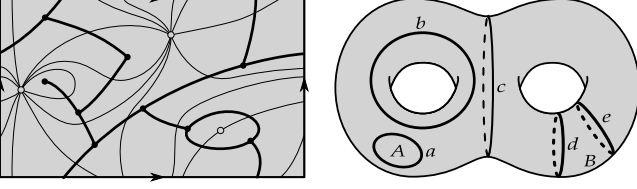FIG. 2: *Left.* A graph (thick lines) in the torus and its dual (weak lines). *Right.* Several cycles in the 2-torus. a is the boundary of A, and c is also homologous to zero because it encloses half of the surface. b, d, e are not homologous to zero. d and e are homologous because they enclose the area B.

follows:

$$\sigma_{\mathbf{u}}\sigma_{\mathbf{v}} = \varphi(\mathbf{u}^t\Omega\mathbf{v})\sigma_{\mathbf{v}}\sigma_{\mathbf{u}} \qquad (2)$$

where

$$\Omega := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad (3)$$

is a $2n \times 2n$ matrix over $\mathbf{Z}_2$ and $\varphi(k) := e^{\pi i k}$, $k \in \mathbf{Z}_2$. We keep the minus sign in Ω because it appears for higher dimensionality or qudits [11]. The group of all the operators generated by the set of σ-operators is the Pauli group $\mathbf{P}_D(n)$.

There exists a nice construction using the Pauli group to construct the symplectic or stabilizer codes [12], [13]. For any subspace $V \subset \mathbf{Z}_2^{2n}$, we define the subspace $\widehat{V} := \{\mathbf{u} \in \mathbf{Z}_2^{2n} \mid \forall \mathbf{v} \in V \; \mathbf{v}^t\Omega\mathbf{u} = 0\}$. Let $V_{\mathcal{C}} \subset \mathbf{Z}_2^{2n}$ be any *isotropic* subspace, that is, one verifying $V_{\mathcal{C}} \subset \widehat{V}_{\mathcal{C}}$ [11]. Let B be one of its basis and set $\mathcal{S} = \{\sigma_{\mathbf{v}} \mid \mathbf{v} \in B\}$. The code

$$\mathcal{C} := \{|\xi\rangle \in \mathcal{H}_2^{\otimes n} \mid \forall \sigma \in \mathcal{S} \; \sigma|\xi\rangle = |\xi\rangle\} \qquad (4)$$

detects the error $\sigma_{\mathbf{u}}$ iff $\mathbf{u} \notin \widehat{V}_{\mathcal{C}} - V_{\mathcal{C}}$ and thus its distance is

$$d(\mathcal{C}) = \min_{\mathbf{u} \in \widehat{V}_C - V_C} |\mathbf{u}|, \qquad (5)$$

where the norm is just the number of qubits on which $\sigma_{\mathbf{u}}$ acts nontrivially. The set $\mathcal{S}$ is the stabilizer of the code. This way, the problem of finding good codes is reduced to the problem of finding good isotropic subspaces $V_{\mathcal{C}}$.

Thus far we have dealt with the purely algebraic structure of codes. Now, we turn to the connection with topology. For a *surface* we understand a compact connected 2-dimensional manifold. Well known examples of surfaces are the sphere S, the torus T and the projective plane P. More generally one can consider the g-torus or sphere with g handles gT, and the sphere with g crosscaps gP. gT and gP are said to have *genus* g. In fact, it is a well known result of surface topology that the previous list of surfaces is complete.

We shall not be restricted to codes based on regular lattices on a torus, or toric codes [7], but we shall use general graphs embedded in surfaces of arbitrary genus in order to explore optimal values of the encoding rate $\frac{k}{n}$. A *graph* Γ is a collection of *vertices* V and *edges* E. Each edge joins two vertices. A graph is usually visualized flattened on the plane. Now consider a graph Γ embedded in a surface M (see Fig. 2). When $M - \Gamma$ is a disjoint collection of discs, we say that the embedding is a *cell embedding*. Let us gather these discs into a set of *faces* F.

We now introduce the notion of a *dual graph*, which is crucial for the construction of quantum topological codes. Given a cell embedding Γ_M of a graph Γ in a surface M, the dual embedding Γ_M^* is constructed as follows. For each face f a point f^* is chosen to serve as a vertex for the new graph Γ^*. For each edge e lying in the boundary of the faces f_1 and f_2, the edge e^* connects f_1^* and f_2^* and crosses e. Each vertex v corresponds to a face $v*$. The idea is illustrated in Fig. 2.

Let us enlarge a bit the concept of surface. Take a surface M and delete the interiors of a finite collection of non-overlapping discs. We say that the resulting space is a *surface with boundary*. Note that for any cell embedding in such a surface, the boundary is a subset of the embedded graph. One could argue that no dual graph can be defined for this surfaces, but in fact this is not a major difficulty. It is enough to think that some of the vertices in the dual graph are 'erased'. For us the most important example of surface with boundary will be the h-holed disc D_h, $h \geq 1$.

Consider a cell embedding Γ_M on a surface, with or without boundary. In order to construct the physical system realizing the topological code $\mathcal{C}$, we attach a qubit to each edge of the graph Γ. The study of the stabilizer and correctable errors of the code gets benefited by using Z_2 homology theory, which we shall now introduce. Consider a Z-type operator $\sigma_{\mathbf{0\lambda}}$. A chain is a formal sum of edges $c_1 = \sum_j \lambda_j e_j$. We relate chains and Z-type operators setting $\sigma_{c_1} := \sigma_{\mathbf{0\lambda}}$. Similarly, given a cochain or formal sum of dual edges $c^1 = \sum_j \lambda'_j e_j^*$, we relate it to an X-type operator setting $\sigma_{c^1} := \sigma_{\mathbf{\lambda'0}}$. There is a natural product between chains and cochains, namely $(c^1, c_1) := \mathbf{\lambda} \cdot \mathbf{\lambda}'$. The point is that

$$\sigma_{c^1}\sigma_{c_1} = \varphi((c^1, c_1))\,\sigma_{c_1}\sigma_{c^1}. \qquad (6)$$

This expression already shows that the commutation relations of operators are determined by the topology of the cell embedding Γ_M. Compare with (2).

Note that a chain is nothing but a collection of edges, those with a coefficient equal to one, and thus can be easily visualized as lines drawn on the surface. If a chain has an even number of edges at every vertex, we call it a *cycle*. When a cycle encloses an area of the surface, we say that it is a *boundary*. If two cycles enclose an area altogether, they are said to be *homologous*. Boundaries are homologous to zero. Figures 2 illustrate these concepts. *Cocycles* are defined analogously but in the dual graph. *Coboundaries* are a bit different, however, at least in the case of surface with boundary. If cutting the sur-

face along a cocycle divides it apart into two pieces, then it is a coboundary. Given a face f, we shall denote by ∂f its boundary. Similarly, given a dual face v^* we denote its coboundary by δv^*, see Fig. 5.

Before stating the main result about general constructions of topological quantum codes for arbitrary graphs embedded in surfaces, we need a pair of ingredients. Given a a surface M there exists always some cell embedding on it. The *Euler characteristic* of M is defined by

$$\chi(M) := |V| - |E| + |F|, \qquad (7)$$

and it does not depend upon the embedded graph. Now, let Γ_M be a cell embedding of a graph Γ in a surface M. We define the distance $\mathrm{d}(\Gamma_M)$ as the minimal length (edge amount) among those cycles which are not homologous to zero. For surfaces with boundary, $\mathrm{d}(\Gamma_M^*)$ should be understood as a symbol denoting the minimal length among cocycles.

Theorem 1 ***Topological codes.*** *Let Γ_M be a cell embedding of a graph in a surface. The symplectic code $\mathcal{C}$ of length $n = |E|$ with stabilizer $\mathcal{S} = \{\,\sigma_{\delta v^*} \,|\, v \in V\,\} \cup \{\,\sigma_{\partial f} \,|\, f \in F\,\}$ has distance $d = \min\{\mathrm{d}(\Gamma_M), \mathrm{d}(\Gamma_M^*)\}$ and encodes $k = 2 - \chi(M)$ qubits if M does not have any boundary or $k = 1 - \chi(M)$ qubits if it does.*

Proof. This proof involves homology theory. Following standard notation, we denote the first homology group by $H_1 = Z_1/B_1$ and the first cohomology group by $H^1 = Z^1/B^1$. Now, since $(\delta v^*, \partial f) = 0$, the space $V_{\mathcal{C}}$ is isotropic. Note that $V_{\mathcal{C}} \simeq B^1 \oplus B_1$. A key observation is that $(\delta v^*, c_1) = 0$ iff $c_1 \in Z_1$, and similarly for ∂f and Z^1. In other words, $\hat{V}_{\mathcal{C}} \simeq Z^1 \oplus Z_1$, and the distance (5) is the one stated. As $\dim \hat{V}_{\mathcal{C}} - \dim V_{\mathcal{C}} = 2k$, where k is the number of encoded qubits, it only remains to know the dimension of the homology and cohomology group. But we have $H_1 \simeq H^1 \simeq \mathbf{Z}_2^{2-\chi}$ for surfaces without boundary and $H_1 \simeq H^1 \simeq \mathbf{Z}_2^{1-\chi}$ for surfaces with boundary. $\square$

Since $\chi(gT) = 2 - 2g$, the g-torus yields codes with $k = 2g$ logical qubits. $\chi(gP) = 2 - g$, and thus codes from gP encode $k = g$ qubits. For the h-holed disc D_h, $\chi(D_h) = 1 - h$ and $k = h$. The parity check matrix H of a topological code has the diagonal form $diag(H_1, H_2)$ where the matrices H_1 and H_2 are in essence the incidence matrices of Γ and Γ^*. Thus, topological codes are an example of generalized CSS codes [14], [15].

Note that uncorrectable errors are related to cycles which are not homologous to zero. This is exemplified as part of Fig. 5. Therefore, the whole problem of constructing good topological codes related to a certain surface relies on finding embeddings of graphs in such a way that both the embedded graph *and* its dual have a big distance whereas the number of edges keeps as small as possible. Thus, we find that this quantum problem can be mapped onto a problem of what is called *extremal topological graph theory*, a branch devoted to graph embeddings on surfaces [16] and the computation of maxima/minima of certain graph properties. To this end, we find very useful to introduce the following concept.

Definition 2 *Given a surface M and a positive integer d, we let the quantity $\mu(M, d)$ be the minimum number of edges among the embeddings of graphs in M giving a code of distance d.*

Since calculating the value of the function μ is a hard problem, we shall investigate some of its properties. The issue of connectivity between sites suggests also the introduction of a refinement of μ. The quantity $\mu_c(M, d)$ is defined as $\mu(M, d)$ but with the restriction that the graphs can have faces with at most c edges and vertices lying in at most c edges. By connectivity here we mean that vertex $\sigma_{\delta v^*}$ and face $\sigma_{\partial f}$ operators should act over at most c qubits. A loosely connected system simplifies the error correction stage.

III. OPTIMAL ENCODING RATES

There is an interesting result in surface topology stating that every surface can be obtained by combination of S, P and T. To perform this connection over two surfaces, say M_1 and M_2, one chooses two discs $D_i \subset M_i$. The *connected sum* of M_1 and M_2, denoted $M_1 \sharp M_2$, is constructed by deleting the interiors of D_1 and D_2 and identifying its boundaries. Connecting $g \geq 0$ tori to a sphere one gets gT, and connecting $g \geq 1$ projective planes one gets gP. The point is that given embeddings of distance d in M_1 and M_2, a new embedding can be constructed in $M_1 \sharp M_2$ in such a way that the number of edges does not increase and the distance is preserved. The procedure is displayed in Fig. 3. This implies the following result, which we shall use to proof asymtoptic properties about minimal sizes of topological codes:

Proposition 3 ***Topological Subadditivity.*** *Given two surfaces M_1 and M_2*

$$\mu(M_1 \sharp M_2, d) \leq \mu(M_1, d) + \mu(M_2, d). \qquad (8)$$

Let us apply these tools starting with the torus T, the simplest orientable surface with nontrivial first homology group. In [7], a family of toric codes was presented, in the form of self-dual regular lattices on the torus. This is a very simple instance of topological graph theory. One can consider other self-dual regular lattices embedded on the torus. All of them share the property that vertex $\sigma_{\delta v^*}$ and face $\sigma_{\partial f}$ operators act on $c = 4$ qubits. Among them, we have found an optimal family of lattices that demand half the number of qubits. Examples of both systems of lattices are depicted in Fig. 4, were the torus is represented as a quotient of the plane through a tessellation. The original toric codes lead to a family of $[[2d^2, 2, d]]$ codes [7]. Our lattices give $[[d^2 + 1, 2, d]]$ codes. This already shows that

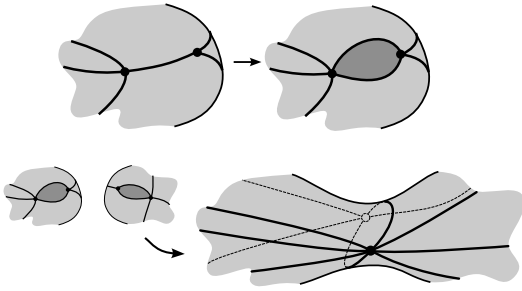$$\mu(T, d) \leq \mu_4(T, d) \leq d^2 + 1. \qquad (9)$$

FIG. 3: The construction that proves the topological subadditivity of μ. The first step is to perform a cut along a selected edge in each of the embeddings to be connected. Then the resulting boundaries must be identified.
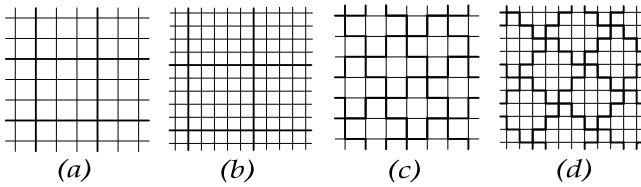

FIG. 4: Four self-dual lattices on the torus. Here the torus is represented as a quotient of the plane and a tessellation of it. Thick lines are the border of tesserae. *(a),(b):* The toric codes introduced in [7], for $d = 3$ and $d = 5$. *(c),(d):* The optimal regular toric codes for $d = 3$ and $d = 5$.

Invoking topological subadditivity, we learn that $\mu(nT, d)$ is $O(d^2)$, that is, it grows at most quadratically with d.

This analysis shows that our toric lattices yield a better encoding rate $\frac{k}{n} \sim \frac{2}{d^2}$ than the original ones with $\frac{k}{n} \sim \frac{1}{d^2}$. However, despite being optimal, these toric lattices produce encoding rates vanishing in the limit of large n qubits (see Fig. 1). Then, a major challenge arises: is it possible to find topological quantum codes with non-vanishing encoding rates? And what about approaching the maximum value of 1? The answer is positive in both cases and we hereby show the construction.

To this end, let us introduce the complete graph K_s as the graph consisting of s vertices and all the possible edges among them. A closer examination of Fig. 4 reveals that the lattice giving a $[[10, 2, 3]]$ code is a self-dual embedding of K_5. This suggests considering self-dual embeddings of K_s, since such an embedding would give a $[[\binom{s}{2}, \binom{s}{2} - 2(s-1), 3]]$ code. In fact, these embeddings are possible in orientable surfaces with the suitable genus as long as $s = 1 \pmod 4$ [16]. In Fig.2 we show the constructions of such an embedding. Due to topological subadditivity and the fact that $\mu(gT, d) \geq \mu(gT, 1) = 2g$, the family of codes given by self-dual embeddings of complete graphs K_s is enough to show that for $d = 3$

$$\lim_{n \to \infty} \frac{k}{n} = \lim_{g \to \infty} \frac{2g}{\mu(gT, 3)} = 1, \qquad (10)$$

that is, the ratio k/n is asymptotically one, and thus good topological codes can be constructed, at least in the case
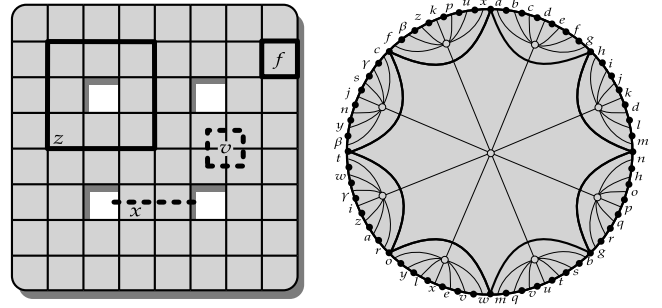

FIG. 5: *Left.* A graph embedding in the surface D_4 yielding a code of distance 3. We display a boundary ∂f, a coboundary δv^* and two uncorrectable errors of Z and X-type: the cycle z is not a boundary, and the cocycle x is not a coboundary. *Right.* The self-dual embedding of K_9 in the 10-torus. Thick lines represent the graph, weak ones its dual.

of codes correcting a single error. Figure 1 displays the rates for this family of codes and also for the optimized toric codes. Now we can appreciate the very different behaviour between toric codes and topological codes embedded in higher genus surfaces, the latter ones allowing us to increase the encoding rate up to its maximal value.

So far we have not touched upon the question of physical implementations. Consider the system of qubits arranged according to a given graph embedding, as explained above. The hamiltonian

$$H = -\sum_{f \in F} \sigma_{\partial f} - \sum_{v \in V} \sigma_{\delta v^*} \qquad (11)$$

has a degenerate ground state whose elements are the protected codewords. This system is naturally protected against errors [7]. A major drawback is the requirement that the physical disposition of the qubits must give rise to a nontrivial topology. It is difficult to imagine an experimentalist constructing a system living in a torus, for example.

However, the formalism that we have presented allows us to use surfaces with boundary. In particular, the h-holed disc D_h gives rise to codes encoding $k = h$ qubits but has the advantage of being a subset of the plane. Figure 5 shows an example of a very regular embedding in D_4. It is apparent how to generalize this example to a higher number of encoded qubits k and distances d. As in the case of surfaces without boundaries, the length of the code will scale as $O(kd^2)$.

IV. CONCLUSIONS

Finally, we want to emphasize that the locality of topological codes is a very important issue for their implementation in physical systems, now more feasible after the introduction of planar topological codes. The embeddings of complete graphs provide encoding rates that overcome the barrier established so far by toric codes.

Moreover, we have introduced a measure μ that establishes an interplay between quantum information and extremal topological graph theory.

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